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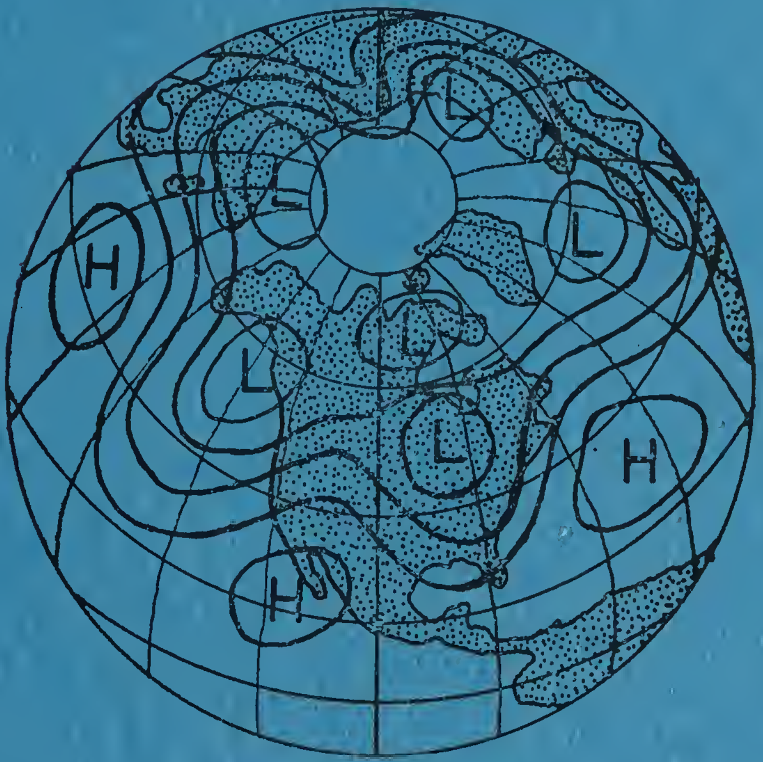
Lamont Geological Observatory
of
Columbia University
PALISADES, NEW YORK

OBJECTIVE WEATHER MAP ANALYSIS BY POLYNOMIAL METHODS

by

RICHARD L. PFEFFER
ALFRED PESCHEL
and
BURTON KAUFMAN



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by
POLYNOMIAL FITTING

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Richard L. Pfeffer

Alfred H. Peschel

Burton Kaufman

Lamont Geological Observatory
Columbia University
Palisades, New York

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1. INTRODUCTION

In order to investigate weather processes, one must be able to determine the physical state of the atmosphere at a given instant of time. There are at present between four and five hundred meteorological stations throughout the northern hemisphere at which temperature, humidity, pressure, and horizontal wind velocity are measured twice daily as a function of altitude. These measurements generally extend to a height of fifteen kilometers or more, and thereby include approximately the lower eighty percent of the mass of the atmosphere. The meteorological stations are distributed horizontally in such a way that over the continents each report represents, on the average, conditions over an area of twenty-five thousand square kilometers. Since most middle and high latitude weather phenomena are determined by motions over several million square kilometers, this sampling meets the minimum requirements for dynamical investigations. Over the oceans, on the other hand, the available reports are so few that each station report must represent conditions over an area close to four million square kilometers.

In order to study the dynamics of the atmosphere, we must be able to estimate the magnitudes of various meteorological parameters at a network of discrete, equally spaced points. This task has been accomplished in the past by careful preparation of synoptic charts, in which isolines are drawn by hand between plotted station reports of pressure, temperature, humidity, wind speed and direction, etc., and visual interpolation between isolines in the neighborhood of each grid point. The considerable time and effort required for the hand preparation of synoptic charts and the subsequent tedious task of extracting grid-point information, needed for example for finite difference calculations, seriously limits the number of meteorological situations which can be

diagnosed within a reasonable period of time. This is especially true if data are required at several levels in the vertical.

In recent years, a number of proposals for objective weather map analysis using high-speed electronic computers have been put forth (see, for example, Panofsky, 1949; Gilchrist and Cressman, 1954; Cressman, 1957, 1959). The proposed methods involve either the averaging of station data or the fitting of geometrical surfaces through these data to obtain grid-point values. The averaging schemes require less computer time but they tend to flatten out maxima and minima in the field (particularly in regions in which an experienced analyst would interpret a more extreme value than any value reported at the neighboring stations), and they contain a source of systematic error due to the irregular density of the observing stations. The surface-fitting schemes require a greater amount of computer time but they do not suffer from the above-mentioned shortcomings. The aim of this paper is to focus more attention on surface fitting as a method of objective weather map analysis for research purposes over regions with comparatively good data coverage. The particular methods to be discussed are those of polynomial surface fitting. In the present paper such methods are applied to the analysis of synoptic-scale weather maps, but it should be noted that they are equally applicable to the analysis of charts representing other scales of motion (e.g., mesoscale).

2. EXACT-FIT POLYNOMIAL METHOD

The exact-fit polynomial method consists in passing an n^{th} degree polynomial surface through the values of a variable at $\frac{1}{2}(n+2)(n+1) \equiv q$ known points and solving the system of q simultaneous linear equations to determine the coefficients of the equation of the surface. The coordinates at any other point may then be substituted into this equation to determine the

value of the variable at that point. This method can be used to fit a single high-degree polynomial surface through all of the station data at any given level, or it can be used to fit many low-degree polynomial surfaces through the data in the neighborhood of the different grid points. In the present investigation we follow the latter procedure and seek the lowest-degree polynomial surface that gives satisfactory results. A plane surface is eliminated as a possibility by virtue of the fact that linear interpolation cannot yield more extreme grid-point values than the station values themselves. Hence, we try the most general second-degree polynomial approximation,

$$\theta = A_0 + A_1 x + A_2 y + A_3 x^2 + A_4 x y + A_5 y^2 \qquad (1)$$

to represent each meteorological variable, θ , in the neighborhood of a grid point, (x, y). Here, x and y are the horizontal coordinates of the grid-point and the A's are the unknown coefficients. The most straightforward approach would be to pass a second degree polynomial surface through the θ's at the six stations nearest to the grid point. However, for certain distributions of station reports it is impossible to obtain a valid solution either because of the occurrence of "ill-conditioned" (or "near singular") matrices, or because of the fact that the grid-point location relative to the station reports is such that the problem becomes one of polynomial extrapolation rather than interpolation. Ill conditioning may be defined as the existence of a near zero denominator in the solution. This phenomenon occurs when the six stations lie on, or near, a conic section (i.e., parabola, hyperbola, or ellipse). An example of such a station distribution is shown in fig. 1. The problem is the same as that which arises when we attempt to pass a plane through three points which are exactly, or nearly, colinear. This, however,

is a separate problem from the one which arises when there is a skewed distribution of station reports relative to the grid point -- that is, when the station reports do not in some way surround the grid point so that the problem is one of extrapolation rather than interpolation. In order to avoid the latter difficulty, we divide the region surrounding each grid point into quadrants and select first the nearest station report in each quadrant. The two remaining points needed to determine the coefficients of the second degree polynomial are selected as the nearest ones to the grid-point which have not already been chosen, regardless of which quadrants they lie in. This insures a more spacially balanced distribution of station values.

The values of x, y and θ at each of the six stations are substituted into (1) to yield six equations in the six unknowns, $A_0, A_1, \ldots A_5$. These equations may be expressed in the following matrix form:

$$\{\theta_i\} = [M_{ij}]\{A_j\} \qquad \begin{matrix}(i = 1, 2, \cdots 6)\\(j = 1, 2, \cdots 6)\end{matrix} \tag{2}$$

where $[M_{ij}]$ is the matrix of the x's and y's. The solution of this equation is

$$\{A_j\} = [M_{ij}]^{-1}\{\theta_i\} \tag{3}$$

The grid-point value of θ is obtained by substituting into equation (1) the x and y coordinates of the grid point and the values, A_j, determined from (3). In practice it is convenient to locate the origin of the x - y coordinate system at the grid point, in which case (1) reduces to $\theta_{GRID} = A_o$.

With this choice, it is necessary to use only the first row of the inverse matrix (M_{ij}) to determine A_o. It follows that the value at any grid point may be expressed in terms of the θ's at the six neighboring stations by

$$\theta_{GRID} = M_1\theta_1 + M_2\theta_2 + M_3\theta_3 + M_4\theta_4 + M_5\theta_5 + M_6\theta_6 \qquad (4)$$

where M_1, M_2, ... M_6 are the elements of the first row of the inverse matrix. It is clear that the sum of the M's must be equal to unity. The M's may thus be interpreted as weights attached to the station values. This interpretation leads to a useful criterion for recognizing ill-conditioning of (2). If (2) is ill-conditioned, θ_{GRID} will be the small difference between large quantities of opposite sign and the computed values of this quantity will not be valid solutions to the problem. The same symptoms will occur if the distribution of stations is skewed relative to the grid point so that the problem becomes one of extrapolation rather than interpolation. A test on the magnitudes of the M's has been constructed to insure that each M is sufficiently small. If the six station values yield M's which satisfy the test, the grid-point value will be obtained by computing θ_{GRID} from (4); if the test is not satisfied, a new sixth station is chosen to replace a previous sixth station until either the test is satisfied and computed or until fifty trials are carried out in this manner. If, after fifty trials, the criterion is not satisfied, a grid-point value can still be computed as an average of a selected number trials meeting certain test requirements. For the first three trials the initial five stations are used repeatedly, and only the sixth station is replaced when necessary. For the fourth trial and thereafter, the sixth and seventh station nearest to the grid point replace the fifth and sixth. Where there are fewer than

fifty stations per map, we must of course limit the number of allowable trials accordingly.

The requirements that the six M's must satisfy are now described. Let $X_m = \max M_i$ ($i=1, 2, \ldots, 6$). If X_m is greater than .8, the value is rejected and a further trial is attempted as described above. If X_m is less than or equal to .8, we check to verify that this weight is not attached to either one of the two farthest stations; this is rarely the case but when it occurs, questionable values for ρ_{GRID} result. Nor should the two most remote stations have a combined weight of greater than or equal to .5. Again, if the latter two requirements involving the two most remote stations are not met, we proceed to a subsequent trial. If, however, these two requirements are satisfied, we check to see that the smallest absolute value of any six elements in the first row of the inverse matrix is less than 0.1; if the latter is not the case, we store A_o and X_m and proceed to the next trial. If, however, there exists one element whose absolute value is smallest and less than 0.1, and if the smallest element is greater than -0.2 then we shall accept the A_o as the computed grid-point value and proceed to the evaluation of the next grid point. If we have fifty unsuccessful attempts, we try to obtain a grid-point value by scanning the stored values of A_o and X_m. If at least seven such A_o's exist, we assign their mean to the grid point and go on to evaluate the next grid-point value. In areas where all the data are on one side of the grid point, it is not possible to obtain values for grid points; these have to be obtained by alternate means.

It should be emphasized that the repeated process of obtaining a new sixth point is solely for the purpose of arriving at a distribution of stations that will lead to a stable matrix; i.e., a matrix for which small changes in

the location of stations relative to each other lead to small changes in the height A_o of the surface directly above the grid point that we wish to evaluate. It is of little importance that we may be using a sixth station thousands of miles away from the grid point in question, since, as stated above, the weight attached to this station value is checked to insure that it is relatively small. In this situation the role of the remote station is to provide a sound distribution for the fitting of a surface, while the role of the nearby stations is to provide a realistic value for the grid point.

A standard matrix inversion technique based upon Gauss-Jordan elimination is used to obtain the inverse of the first matrix. For each ~~necessary~~ subsequent inverse required, a completion scheme, after Bodewig (1959), is employed, which yields the desired inverse with a minimum amount of computation using the first matrix, its inverse and the matrix under consideration. It should be pointed out that each of the possible 50 matrices that may be required differs from the first only in its sixth row.

The test on the first row elements of the inverse matrix is invariant under the multiplication of the station coordinates by a constant, and is thus independent of any particular mesh conditions. Moreover, since the exact surface fit recognizes unfavorable distributions of station values, a gridpoint value computed from a stable matrix is certain to be representative of the data surrounding it.

3. LEAST-SQUARES POLYNOMIAL METHOD

The exact-fit polynomial method described in the preceding section requires six independent observations to define a second-degree polynomial surface for the determination of each grid-point value of $\mathcal{G}$. In order to circumvent the problems which arise due to ill conditioning, or to a skewed distribution

of reporting stations relative to the grid point, it is sometimes necessary to carry out a substantial number of trial matrix inversions until certain empirically derived criteria are met. An alternate procedure for avoiding these difficulties would be a least square second-degree polynomial fit through more than six station values of θ .

Experience has shown that a least square approximation with surfaces of the form, (1), does not lead to a near-singular matrix equation defining the coefficients of the solution surface, provided that at least twelve independent observations enter into the calculations (Cressman, 1957). Unfortunately, however, the greater number of stations thus needed introduce the additional problem of smoothing. It is important to note that the exact fit method requires the minimum analysis area for an approximation with surfaces of the type, (1), and thus entails a minimum amount of smoothing. The latter remark leads to the desired goal of modifying the conventional least square approach so that the polynomial surfaces fit more closely at the nearby stations than at the more remote ones. The more remote stations then serve the same object of insuring a non-singular matrix solution. In this manner smoothing problems can be greatly reduced.

In general let us choose N independent station values in the same manner as described in the exact fit method; N must be greater or equal to twelve, assuring non-singularity. Let x_i , y_i $(i = 1, \ldots, N)$ be the coordinates of such stations, again transformed so that the grid point is at the origin. We seek to compute the coefficients of the least square surfaces of the form, (1). In the conventional least square approach we minimize the non-negative function of six variables

$$S(A_0, A_1, A_2, A_3, A_4, A_5) = \sum_{i=1}^{N} (\theta_i - A_0 - A_1 x_i - A_2 y_i - A_3 x_i^2 - A_4 x_i y_i - A_5 y_i^2)^2 \qquad (5)$$

by setting

$$\frac{\partial S}{\partial A_j} = 0 \qquad j = 0, \cdots, 5 \tag{6}$$

Equation (6) is the associated matrix equation whose solution, $(A_0, \ldots, A_5)$, yields the coefficients of the desired least square surface; again, the constant term, A_0, is the grid-point value desired. Let

$$D_i = (\theta_i - A_0 - A_1 x_i - A_2 y_i - A_3 x_i^2 - A_4 x_i y_i - A_5 y_i^2) \tag{7}$$

be the deviation of the observed value, θ_i, from the least square surface

$$A_0 + A_1 x + A_2 y + A_3 x^2 + A_4 x y + A_5 y^2 \tag{8}$$

at the i^{th} station $(i = 1, \ldots, N)$. Since we wish to make D_i small for those stations nearby, each deviation will not be given an equal weight. Instead, the deviations corresponding to nearby stations will be given a greater weight than those corresponding to far away stations. We shall in fact assign N weights, W_i $(i = 1, .., N)$, to each deviation by defining

$$\omega_i = \rho / \rho_i \tag{9}$$

where ρ and ρ_i are the distances between the grid point and the nearest station, and between grid point and the i^{th} station, respectively. Then we minimize, as before,

$$S(a_0, a_1, a_2, a_3, a_4, a_5) = \sum_{i=1}^{N} \omega_i D_i^2 \tag{10}$$

obtaining the associated linear system of equations

$$\frac{\partial S}{\partial a_j} = 0 \qquad j = 0, \cdots, 5 \tag{11}$$

If, for example, all stations are approximately equidistant from the grid point, the weights are nearly the same and equal to one, yielding a conventional least square parabolic fit. If, however, one station is significantly closer to the grid point than the remaining ones, the least square surface fits more exactly at that point and the grid-point value will be more influenced by that station. Consequently, much undesirable smoothing is either reduced or eliminated.

Attention is now focused upon the problem of formulating criteria which will enable us to decide whether the relative positions of the stations with respect to the grid point are favorable in terms of the extrapolation difficulties mentioned earlier. Let us now consider the N stations as mass points with mass equal to one; it is then possible to interpret the barycenter of such a system of data points as its center of gravity. The barycenter has the coordinates

$$\bar{x} = \left(\sum_{i=1}^{N} x_i\right)/N \quad , \qquad \bar{y} = \left(\sum_{i=1}^{N} y_i\right)/N \tag{12}$$

and in the neighborhood of such a point $(\bar{x}, \bar{y})$, one can expect reasonable values by polynomial surface fitting, since the data must surround this point adequately. Therefore, it is reasonable to compute only those grid points by the above weighted least square method if the distance between the barycenter and the grid point is small. For this purpose, the following procedure is suggested. During the first pass over the grid point field, only those grid points are computed whose associated N data points define a barycenter which

lies within 0.6 grid units from the grid point. In the first pass only original station data are used. In subsequent passes computed grid point information obtained in the previous pass will be employed. Thus, the second pass evaluates grid-point values from original data plus data computed during the first pass only. The allowable distance between barycenter and grid point for the second, third and fourth pass are respectively 0.8, 1.0, and 1.1.

4. COMPARISON WITH OTHER METHODS OF ANALYSIS

In order to secure some measure of the usefulness of the polynomial fitting methods described in the last two sections, it was felt desirable to compare analyses based on these methods with those made by hand and also with those made by another method of objective analysis based on averaging procedures. The authors have succeeded in modifying an averaging scheme due to Cressman (1959) so as to make it independent of an initial forecast. Let $P_o(x, y)$ denote an initial guess at a grid point with coordinates x and y and $Q(x_i, y_i)$ the value at the i^{th} reporting station with coordinates (x_i, y_i), which is situated within N grid lengths from (x, y). $P_o(x, y)$ may be set initially to the mean of the reporting station values or to that of the nearest station value; it does not matter which. The expression for the m^{th} approximation to the grid-point value may be written in the form

$$P_m^N(x\ y) = P_{m-1}^N(x, y) + \frac{1}{n}\sum_{i=1}^{n}\frac{N^2 - D_i^2}{N^2 + D_i^2}\left[Q(x_i, y_i) - P_{m-1}^{N-1}(x, y)\right] \tag{13}$$

where D_i is the distance between the grid point and the i^{th} reporting station; N assumes the successive values 5, 4, 3, 2 and 1, and n is the number of reporting stations within N of (x, y). An iteration is said to

be complete when N equals 1, and the values at each iteration form a converging sequence. Thus, the initial values assigned to P_0 (x, y) cannot affect the final grid-point value if the procedure is cut off at the end of an iteration which differs from the proceding one by an approximately small amount.

Charts of winds and potential temperatures over North America at 14 elevations from the ground to 120 mb were prepared by the three methods of objective analysis and by hand for 16 March, 1956 at 0300 and 1500 Z. Grid-point values of each of these variables were obtained at 320 points spaced 225 km apart and a computer program was written for contouring the resulting field of values. Approximately 90 percent of the 320 grid-point values can be calculated by polynomial methods. Near the boundaries of the grid system, where the distribution of station values makes it impossible to obtain a valid polynomial solution, the grid-point values were determined by divided differences in the case of the exact-fit polynomial analysis and by fitting a least square plane to the adjoining grid-point and station data in the case of the least square polynomial analysis. The results near the boundaries are therefore not directly comparable. Unfortunately, the statistics shown in the tables to be presented shortly are also contaminated by these values near the boundary.

Figs. 2, 3 and 4 show comparisons among the four methods of analysis of the potential temperature field at 700, 500 and 150 mb, respectively, at 0300 Z on 16 March 1956. At all three levels, the averaging method suppresses the extreme values more than either polynomial method. By and large the two polynomial methods give results which agree more closely with the hand analyses than does the averaging method.The two polynomial solutions for the same three levels for 1500 Z are compared in figs. 5, 6 and 7 with the hand analyses.

Tables 1, 2, 3 give means and variances of the winds and temperatures as analyzed by each method, and sums of squares of the differences between the hand analysis and each method of objective analysis. Figures 8, 9, 10 and 11 give the vertical velocity computed from solutions of the dynamical ω -equation (where ω is the total time rate of change of pressure on a parcel of air) using each of the methods of temperature and wind analysis. The averaging method is seen to suppress the extreme values more than either of the two polynomial methods.

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